ON EXPLICIT PROBABILITY DENSITIES ASSOCIATED WITH FUSS-CATALAN NUMBERS

DANG-ZHENG LIU, CHUNWEI SONG AND ZHENG-DONG WANG

Abstract. In this note we give explicitly a family of probability densities, the moments of which are Fuss-Catalan numbers. The densities appear naturally in random matrices, free probability and other contexts.

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In this note we study a family of probability densities π_s, $s \in \mathbb{N}$, which are uniquely determined by the moment sequences $\{m_0, m_1, \ldots, m_k, \ldots\}$ [1]. Here

$$m_k = \frac{1}{sk+1}\binom{sk+k}{k} \tag{1}$$

are known as Fuss-Catalan numbers in Free Probability Theory [6]. The densities π_s belong to the class of Free Bessel Laws [3] and are known to appear in several different contexts, for instance, random matrices [1, 3, 6], random quantum states [4], free probability and quantum groups [3]. More precisely, π_s is the limit spectral distribution of random matrices in the forms like $X_1^s X_1^{*s}$ and $X_1 \cdots X_s X_s^* \cdots X_1^*$, where $X_1, \ldots, X_s$ are independent $N \times N$ random matrices; in free probability we have the free convolution relation: $\pi_s = \pi_1^{\boxtimes s}$ [6, 3].

More generally, T. Banica et al [3] introduce a remarkable family of probability distributions $\pi_{s,t}$ with $(s,t) \in (0,\infty) \times (0,\infty) - (0,1) \times (1,\infty)$, called free Bessel laws. π_s is the special case where $t = 1$, i.e., $\pi_{s,1} = \pi_s$. The moments of $\pi_{s,t}$ are the Fuss-Catalan polynomials (Theorem 5.2, [3]):

$$m_k(t) = \sum_{j=1}^{k} \frac{1}{j}\binom{k-1}{j-1}\binom{sk}{j-1} t^j. \tag{2}$$

Indeed, the following relation holds [3]:

$$\pi_{s,1} = \pi_1^{\boxtimes s}, \quad \pi_{1,t} = \pi_1^{\boxplus t}. \tag{3}$$

The distribution $\pi_{1,t}$, the famous Marchenko-Pastur law of parameter t or called free Poisson law, owns an explicit formula:

$$\pi_{1,t} = \max(1-t, 0)\delta_0 + \frac{\sqrt{4t - (x-1-t)^2}}{2\pi x}. \tag{4}$$

In special case,

$$\pi_1 = \pi_{1,1} = \frac{1}{2\pi}\sqrt{4x^{-1} - 1}. \tag{5}$$

Another special case of $\pi_{s,t}$ where an explicit density formula is available is, due to Penson and Solomon [7],

$$\pi_2 = \pi_{2,1} = \frac{\sqrt[3]{2}\sqrt{3}}{12\pi} \frac{\sqrt[3]{2}(27+3\sqrt{81-12x})^{2/3} - 6\sqrt[3]{x}}{x^{2/3}(27+3\sqrt{81-12x})^{1/3}} \, 1_{(0,27/4]}(x). \tag{6}$$

To the best of our knowledge, except for the above special cases there are no explicit formulae available for the other $\pi_{s,t}$. The aim of this work is to give explicit densities of $\pi_s = \pi_{s,1}$, $s \in \mathbb{N}$. We state our main result as follows and its proof is based on the method: how to find an explicit density from a given certain moment sequence used in [8, 5].

Theorem 1. *Let π_s, $s \in \mathbb{N}$ be the unique densities determined by the Fuss-Catalan numbers in Eq.(1), then we have the following formulae*

$$\pi_s(x) = \frac{1_{(0,K]}(x)}{B(\frac{1}{2}, \frac{1}{2}+\frac{1}{s})} \int_{[0,1]^s} \frac{\big(\tau K - x\big)^{1/s-1/2}}{\sqrt{x}\,\big(\tau K\big)^{1/s}} F(t_1,\dots,t_s)\, 1_{\{\tau K \geq x\}} d^s t, \tag{7}$$

where $K = (s+1)^{s+1}/s^s$, $\tau = \prod_{j=1}^{s} t_j$ and $F(t_1) = \delta(t_1 - 1)$ while for $s > 1$

$$F(t_1,\dots,t_s) = \frac{1}{B(\frac{1}{s+1}, \frac{s-1}{2s+2}) \prod_{j=2}^{s} B(\frac{j}{s+1}, \frac{j}{s(s+1)})} \times$$

$$t_1^{\frac{1}{s+1}-1}(1-t_1)^{\frac{1}{2s+2}-1} \prod_{j=2}^{s} t_j^{\frac{j}{s+1}-1}(1-t_j)^{\frac{j}{s(s+1)}-1}. \tag{8}$$

Proof. First, we derive a family of symmetric distributions σ_s, the $2k$-moments of which are m_k in Eq.(1).

Consider the characteristic function of σ_s as follows:

$$\int_{-\infty}^{+\infty} e^{i\xi x} \sigma_s(x) dx = \sum_{k=0}^{\infty} \frac{(-\xi^2)^k}{(2k)!} m_k = \sum_{k=0}^{\infty} \beta_k \frac{(-\xi^2)^k}{k!}, \tag{9}$$

where

$$\beta_k = \frac{1}{sk+1} \binom{sk+k}{k} \frac{k!}{(2k)!} = \frac{1}{sk+1} \frac{(sk+k)!}{(sk)!(2k)!}. \tag{10}$$

A direct computation shows that the ratio

$$\begin{aligned}
\frac{\beta_{k+1}}{\beta_k} &= \frac{sk+1}{s(k+1)+1} \frac{1}{(2k+1)(2k+2)} \frac{\big((s+1)k+1\big)\big((s+1)k+2\big)\cdots\big((s+1)k+s+1\big)}{(sk+1)(sk+2)\cdots(sk+s)} \\
&= \frac{K}{4} \frac{(k+\frac{1}{s+1})(k+\frac{2}{s+1})\cdots(k+\frac{s}{s+1})}{(k+\frac{1}{2})(k+\frac{2}{s})\cdots(k+\frac{s}{s})} \frac{1}{k+1+\frac{1}{s}} \\
&\doteq \frac{K}{4} \frac{(k+a_1)(k+a_2)\cdots(k+a_s)}{(k+b_1)(k+b_2)\cdots(k+b_s)} \frac{1}{k+b_{s+1}}.
\end{aligned}$$

Here

$$b_1 = \frac{1}{2},\ a_i = \frac{i}{s+1} \text{ and } b_{i+1} = \frac{i+1}{s} \text{ for } i = 1, 2, \dots, s. \tag{11}$$

Therefore, using the generalized hypergeometric function, we rewrite

$$\int_{-\infty}^{+\infty} e^{i\xi x}\sigma_s(x)dx = {}_sF_{s+1}(a_1,\ldots,a_s;b_1,\ldots,b_s,b_{s+1};-\frac{K}{4}\xi^2) \tag{12}$$

$$= \int_{[0,1]^s} F(t_1,\ldots,t_s)\,{}_0F_1(b_{s+1};-\frac{\tau K}{4}\xi^2)\,d^st. \tag{13}$$

Note that $a_1 = b_1$ for $s = 1$ but $b_j > a_j > 0, j = 1, 2, \ldots, s$ when $s > 1$, in Eq.(13) we have made use of Euler's integral representation of the generalized hypergeometric function [2].

Next, with the help of the integral representation of Bessel function of the first kind [2], that is, for $\alpha > -\frac{1}{2}$,

$$\begin{aligned} J_\alpha(z) &= \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}\,{}_0F_1(\alpha+1;-\frac{1}{4}z^2) \\ &= \frac{(z/2)^\alpha}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{-1}^{1} e^{izx}(1-x^2)^{\alpha-\frac{1}{2}}dx, \end{aligned} \tag{14}$$

we get

$$\begin{aligned} {}_0F_1(\alpha+1;-\frac{1}{4}z^2) &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{-1}^{1} e^{izx}(1-x^2)^{\alpha-\frac{1}{2}}dx \\ &= \frac{1}{B(\frac{1}{2},\alpha+\frac{1}{2})}\int_{-1}^{1} e^{izx}(1-x^2)^{\alpha-\frac{1}{2}}dx. \end{aligned} \tag{15}$$

Set $\alpha = 1/s$, $z = \sqrt{\tau K}\xi$, we have

$$\begin{aligned} {}_0F_1(b_{s+1};-\frac{\tau K}{4}\xi^2) &= \frac{1}{B(\frac{1}{2},\frac{1}{2}+\frac{1}{s})}\int_{-1}^{1} e^{i\sqrt{\tau K}\xi x}(1-x^2)^{\frac{1}{s}-\frac{1}{2}}dx \\ &= \frac{1}{B(\frac{1}{2},\frac{1}{2}+\frac{1}{s})}\int_{-\sqrt{\tau K}}^{\sqrt{\tau K}} e^{i\xi x}\frac{(\tau K-x^2)^{\frac{1}{s}-\frac{1}{2}}}{(\tau K)^{\frac{1}{s}}}dx. \end{aligned} \tag{16}$$

Combining (16) and (13), after interchanging the order of integration, we then obtain

$$\sigma_s(x) = \frac{1}{B(\frac{1}{2},\frac{1}{2}+\frac{1}{s})}\int_{[0,1]^s} \frac{\left(\tau K - x^2\right)^{1/s-1/2}}{\left(\tau K\right)^{1/s}} F(t_1,\ldots,t_s)\,1_{\{\tau K\geq x^2\}}d^st. \tag{17}$$

Notice the fact

$$\pi_s(x) = \frac{\sigma_s(\sqrt{x})}{\sqrt{x}}1_{(0,\infty)}, \tag{18}$$

the proof is then complete. □

At the end we remark that some properties of the density $\pi_s(x)$ follow easily from Theorem 1, for instance, there are no atoms; the support is $[0, K]$; the density is analytic on $(0, K)$ (Theorem 2.1, [3]).

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School of Mathematical Sciences, LMAM, Peking University, Beijing, 100871, P.R. China

E-mail address: `dzliumath@gmail.com`

E-mail address: `csong@math.pku.edu.cn`

E-mail address: `zdwang@pku.edu.cn`